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MISSILE AT INTERMEDIATE RANGE.

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Introduction. The problem studied in this paper is that of
obtaining optimum trajectories, such as the one for effecting
interception of a target following a known course above the sen-
sible atmosphere, with minimum fuel consumption, or in minimum
time. A procedure is given for determining the trajectory on a
digital computer, and some conditions are given for the corres-
ponding optimum trajectories. The methods apply generally to
problems wherein the range and time are not too great and the
terminal velocity is irrelevant.

An essential feature is in the use of the adjoint system of
differential equations as defined by G. A. Bliss. A somewhat
novel feature is in the use of direct methods and an optimizing
principle in determining the trajectories. A simplification of
the differential equations is effected by using the acceleration
rather than the mass as a variable.

The steering equation, which is well known, is a simple con-
sequence of the optimum principle in control theory which has
received considerable attention in this country and in Russia re-
cently. The throttling relation, also obtained from an optimum
principle, is apparently not generally known: in problems where
the energy available is limited, the energy input is to occur

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when a vector defined by the adjoint system of differential
equations is a maximum in magnitude. These principles reduce
the problem to the solution of a three-point boundary problem.

The following simplifying assumptions are made. The action
takes place above the atmosphere so that aerodynamic forces are
negligible. The trajectory of the missile is assumed to lie in
a plane containing the center of the earth so that the trajectory
is two dimensional, to reduce the number of variables.

1. Basic equations. The equation of motion of a rocket in a
gravitational field, subject to no outside forces, may be writ-
ten as

[1] $\ddot{\vec{r}} = \vec{g} + \vec{a}$,

where $\vec{r}$ is the position vector, $\vec{g}$ is the acceleration due to
gravity, $\vec{a}$ is the acceleration due to thrust, and a dot (°) over
a variable indicates its time derivative. We may write

[2] $\vec{a} = c'\dot{m}\vec{e}/(1-m)$,

where m is the ratio of the mass of fuel which has been consumed
to the initial gross mass of the rocket $\vec{e}$ is the unit vector
in the direction of the thrust, and c' is a constant.

A useful kinematic relation connecting the acceleration and
the fuel consumption is the following

[3] $\int_0^t a\,dt = c'\int_0^t \frac{\dot{m}}{1-m}dt = -c'\ln(1-m)$.

Since the fuel consumed is proportional to m and $\ln(1-m)$ is a
monotonic function of m, conditions involving the final value of
the mass may be rephrased in terms of the integral of the accelera-
tion, subject to the constraints on the size of $\dot{m}$. For practical
purposes we may take $\dot{m}$ as bounded above by some constant $\dot{m}_{max}$

and bounded below by zero, and it may be chosen anywhere on or between these limits, so long as any fuel remains.

The equations of motion then have the form

[4] $\begin{cases} \ddot{x} = g_1 + a\cos p \\ \ddot{y} = g_2 + a\sin p , \end{cases}$

where we may think of g_1 as $-c_g x/r^3$, with c_g a constant and $r^2 = x^2 + y^2$, and g_2 as $-c_g y/r^3$; these equations may be refined or simplified.

To get the needed formulas, let us multiply equations [4] through by two new variables u, v, which are unspecified so far, and integrate formally to get

[5] $\int_0^T [u(\ddot{x} - g_1 - a\cos p) + v(\ddot{y} - g_2 - a\sin p)]dt = 0 ,$

where x, y are any solutions to [4].

Let us consider also a neighboring path whereon the original equations are satisfied. For the first variations, equation [5] becomes

[6] $\int_0^T [u(\delta\ddot{x} - g_{1_x}\delta x - g_{1_y}\delta y - \delta a\cos p + a\,\delta p\sin p)$

$+ v(\delta\ddot{y} - g_{2_x}\delta x - g_{2_y}\delta y - \delta a\sin p - a\,\delta p\cos p)]dt = 0;$

subscripts denote partial derivatives $g_{1_x} = \partial g_1/\partial x$, etc. If we integrate by parts, this may be rewritten

[7] $[u\delta\dot{x} - \dot{u}\delta x + v\delta\dot{y} - \dot{v}\delta y]_0^T + [a(u\cos p + v\sin p)]_{t_1-}^{t_1+}\,\delta t_1$

$+ \int_0^T [\delta x(\ddot{u} - g_{1_x}u - g_{2_x}v) + \delta y(\ddot{v} - g_{1_y}u - g_{2_y}v)$

$- \delta a(u\cos p + v\sin p)$

$- a\,\delta p(-u\sin p + v\cos p)]dt = 0;$

here t_1 is a symbol for any time or times when $\vec{a}$ is

discontinuous.

Now, to simplify equation [7], let us choose u,v as solutions to the system of differential equations

[8] $\begin{cases} \overset{\circ\circ}{u} - g_{1_x}u - g_{2_x}v = 0 \\ \overset{\circ\circ}{v} - g_{1_y}u - g_{2_y}v = 0 ; \end{cases}$

this choice of u,v eliminates the variations of the dependent variables x,y from the integral in [7]. Equations [8] are called <u>adjoint</u> to the variational equations corresponding to [4]; in general, it is an integration by parts and setting to zero the coefficients of terms involving the dependent variables which determines the adjoint system. Let us further choose the solutions so that u(T) = v(T) = 0. Equation [7] is then a fairly general relation connecting the variations of the end values of x,y with the variations of the control variations a,p for any particular fixed value T and any particular trajectory.

2. Variations of dependent variables. If we further choose the solution u_1,v_1 to [8] such that

[9] $\begin{cases} \overset{\circ}{u}_1 = -1 \\ \overset{\circ}{v}_1 = 0, \end{cases}$

and if all of the initial values are given, equation [7] becomes

[10] $\delta x(T) = -[a(u_1 \cos p + v_1 \sin p)]_{t_{1-}}^{t_{1+}} \delta t_1 +$

$+ \int_0^T [\delta a(u_1 \cos p + v_1 \sin p) + a \,\delta p(-u_1 \sin p + v_1 \cos p)]dt;$

similarly, if $\overset{\circ}{u}_2(T) = 0$, $\overset{\circ}{v}_2(T) = -1$,

[11] $\delta y(T) = -[a(u_2 \cos p + v_2 \sin p)]_{t_{1-}}^{t_{1+}} \delta t_1 +$

$$+ \int_0^T [\delta a(u_2 \cos p + v_2 \sin p) + a\, \delta p(-u_2 \sin p + v_2 \cos p)]dt.$$

These are the essential equations for variation and control, since they express the effect on the terminal values of x, y of small changes in the control variables a, p.

It will be necessary to have a fundamental set of solutions to equations [8]; we may choose, to be definite, $u^1, v^1, u^2, v^2,$ $u^3, v^3, u^4, v^4,$ such that

[12]
$$\begin{cases} u^1(0) = 1, \ v^1(0) = 0, \ \mathring{u}^1(0) = 0, \ \mathring{v}^1(0) = 0 \\ u^2(0) = 0, \ v^2(0) = 1, \ \mathring{u}^2(0) = 0, \ \mathring{v}^2(0) = 0 \\ u^3(0) = 0, \cdots \\ u^4(0) = 0, \cdots \qquad\qquad\qquad \mathring{v}^4(0) = 1 \ . \end{cases}$$

Then, for any particular trajectory and any time T, every solution is a linear combination of these, and we may write

[13]
$$\begin{cases} u_i = \sum_1^4 c_{ij} u^j \\ \qquad\qquad i = 1,2 \\ v_i = \sum_1^4 c_{ij} v^j \ , \end{cases}$$

for the proper choice of the c's.

3. Maximizing principle. It is a property of extremals in normal problems that they furnish a maximum to an integral and that this is done by maximizing the integrand; this may be taken as the characteristic of an extremal as follows.

Let us write [5] in the alternate form

[14] $[u\mathring{x} - \mathring{u}x + v\mathring{y} - \mathring{v}y]_0^T + \int_0^T (x\mathring{u} - ug_1 + y\mathring{v} - vg_2)dt$

$$= \int_0^T \vec{a} \cdot \vec{w} \ dt \ .$$

Now suppose we have selected $\vec{w} = u\vec{i} + v\vec{j}$ in some way, where

u,v satisfy [8] and $w = |\vec{w}(t)|$ is a diminishing function of

t with $w(T) = 0$. Without regard for a reason, let us consider

the problem of maximizing the integral on the right in [14], by

choosing $\vec{a}$ properly. It most satisfy the constraints that (i)

$0 \le a \le a_{max}$, where a_{max} is a known function of t, and (ii)

[15] $\int_0^T a\ dt \le C$, a given constant;

C is by [3] a measure of the fuel consumption. For a_{max} we may

take the function

[16] $a_{max} = lesser[c'\mathring{m}_{max}/(1-\mathring{m}_{max}t),\ c\mathring{m}_{max}/(1-m_{max})]$,

where $m_{max}, \mathring{m}_{max}$ are the maximum values of $m, \mathring{m}$, determined

by the rocket mechanism.

We see that to maximize the integral in [14] it is neces-

sary and sufficient that (i) $\vec{a}$ is parallel to $\vec{w}$, so that

[17] $\tan p = v/u$

(the proper branch of p chosen) and (ii)

[18] $a = \begin{cases} a_{max}, & 0 \le t \le t_1 \\ 0, & t_1 \le t \le T, \end{cases}$

t_1 being chosen so that equation [15] is satisfied.

An outline of the proof is given here; it is given in

detail in (1, section 5). Consider the path defined above and

any other path; let the functions for a,p be denoted by A,P

on the second path. A must satisfy the constraints (i),(ii)

specified on a above, so that

[19] $\int_0^T A\ dt \le C$.

It is seen that

$$\int_0^T (\vec{a}-\vec{A}) \cdot \vec{w} \ dt = \int_0^{t_1} (a_{max}-A) w \ dt + \int_0^{t_1} A(1-\cos[p-P]) w \ dt$$

$$- \int_{t_1}^T A \ w \ \cos(p-P) \ dt \ .$$

The sum of the first and third terms is positive or zero and the second term is positive or zero. This establishes that the path just described maximizes the integral.

4. Attaining a fixed point in specified time with minimum fuel. Let us consider, as an example, the problem of determining a trajectory to attain a specified point X_f, Y_f in a specified time T with a minimum of fuel, assuming that all of the initial conditions are given. We will first take up a routine for finding the trajectory and then show how it satisfies certain conditions which can be checked.

Let us guess initial values for the solutions u_1, v_1 and u_2, v_2, solutions to [8], and a linear combination of these

[20] $\begin{cases} u = u_1 \cos \Theta + u_2 \sin \Theta \\ v = v_1 \cos \Theta + v_2 \sin \Theta, \end{cases}$

where Θ is a number to be found. Each of the quantities should have an iteration number, since they will be in an iterative computational routine. Now let

[17] $\tan p = u/v,$

properly chosen. Let us guess also t_1; maximum thrust is to be applied until time t_1, and thrust is zero thereafter.

Now compute the corresponding trajectory. It will lead to terminal values $x(T), y(T)$, which are in error by amounts $X_f-x(T), Y_f-y(T)$. Compute simultaneously a fundamental set of solutions u^i, v^i to the adjoint system of differential equations [8], and two further integrals which occur later in equation [22].

From [17],[20], for fixed or determined u_1, v_1, u_2, v_2, we get

[21] $\delta p = \delta\Theta(u_1 v_2 - u_2 v_1)/([u]^2 + [v]^2).$

The relations for the variations of the end values then become

[22]
$$\delta x(T) = [a(u_1 \cos p + v_1 \sin p)]_{t_1 -} \delta t_1$$
$$+ \int_0^{t_1} \frac{a(-u_1 \sin p + v_1 \cos p)(u_1 v_2 - u_2 v_1)}{[u]^2 + [v]^2} dt\ \delta\Theta$$
$$\delta y(T) = [a(u_2 \cos p + v_2 \sin p)]_{t_1 -} \delta t_1 + \int \ldots .$$

Now set

[23]
$$\delta x(T) = X_f - x(T)$$
$$\delta y(T) = Y_f - y(T),$$

substitute into [22], and solve for $\delta t_1, \delta\Theta$. This gives corrected values of t_1, Θ for the next round.

From the fundamental set of solutions $u^1, \ldots v^4$, corrected estimates are made for the initial values of u_1, v_1, u_2, v_2 and their derivatives. All of the initial values are then available for starting the next iteration.

This routine is continued until some convergence criterion is satisfied: for example one might require

[24] $E = (X_f - x[T])^2 + (Y_f - y[T])^2$

to be less than some preassigned number.

If the routine given above converges, we have found a trajectory which effects at least a stationary value for the fuel consumption compared with paths in some neighborhood. For if we consider neighboring paths in equation [14], we see that the first variation of the left side vanishes, by virtue of equations [8] and the fact that $\delta x(0), \delta y(0), \delta\dot{x}(0), \delta\dot{y}(0), \delta x(T),$ $\delta y(T), u(T), v(T)$ all vanish. But the right side of the equation

is positive for every admissible variation of the acceleration,
as was shown in section 3. Hence there is no neighboring path
whose first variations do not vanish which requires the same
or less fuel. If gravity can be approximated as a linear func-
tion of position, this also establishes the sufficient conditions
for an absolute minimum [see (1), section 5].

5. Variations of terminal time; minimum fuel consumption. Now
suppose that the terminal time T may vary. The change in the
end values of x,y are

$$[25] \qquad \begin{cases} \delta X = \mathring{x}(T)\delta T + \delta x(T) \\ \delta Y = \mathring{y}(T)\delta T + \delta y(T) \ . \end{cases}$$

There are corresponding equations for the terminal values of
u_1, etc.,

$$u_1(T) = -\mathring{u}_1(T)\delta T, \quad v_1(T) = -\mathring{v}_1(T)\delta T, \quad \mathring{u}_1(T) = -1 -\mathring{\mathring{u}}_1(T)\delta T,$$

etc.; in these equations T should have an iteration subscript,
being the approximation associated with the last trajectory run.

Since there are three variables t_1, T, Θ, and only two equa-
tions for X_f, Y_f, a further relation is required. If we are in-
terested in minimum fuel consumption, then t_1 must be a minimum.
Equation [7] may be rewritten

$$[26] \qquad (-\mathring{u}\delta x - \mathring{v}\delta y)_T = [a(u \cos p + v \sin p)]_{t_1-}\delta t_1 \ ,$$

if terms that must vanish are omitted.

If we consider two neighboring trajectories, each of which
effects interception of a target whose coordinates are $X_f(T)$,
$Y_f(T)$, then $\delta x(T) = (\mathring{X}_f - \mathring{x})\delta T$, $\delta y(T) = (\mathring{Y}_f - \mathring{y})\delta T$, and [26]
becomes

$$[27] \qquad (-\mathring{\vec{w}})_T \cdot [(\mathring{X}_f - \mathring{x})\vec{I} + (\mathring{Y}_f - \mathring{y})\vec{J}]_T \delta T = (aw)_{t_1-}\delta t_1 .$$

Hence, for minimum fuel consumption,

$$[28] \qquad \overset{\circ}{\vec{w}}(T) \cdot [(\overset{\circ}{X}_f - \overset{\circ}{x})\vec{i} + (\overset{\circ}{Y}_f - \overset{\circ}{y})\vec{j}]_T = 0:$$

these two vectors must be perpendicular; apparently this is
a fairly general condition associated with minimum fuel con-
sumption. This condition allows a direct determination of the
angle Θ. This in turn defines $\delta\Theta$ as the difference of the
two successive values of Θ.

The value for $\delta x(T)$ which must be used in the iterative
routine is $\delta x(T) = X_f(T) - x(T) + (\overset{\circ}{X}_f - \overset{\circ}{x})_T \delta T$, and the equa-
tions for corrections to T, t_1 are

$$[29] \qquad \begin{cases} X_f(T) - x(T) = -(\overset{\circ}{X}_f - \overset{\circ}{x})\delta T + \delta x(T) \\ Y_f(T) - y(T) = -(\overset{\circ}{Y}_f - \overset{\circ}{y})\delta T + \delta y(T), \end{cases}$$

where $\delta x(T), \delta y(T)$, linear in $\delta\Theta, \delta t_1$, are given by equations
[22]. Since $\delta\Theta$ is already determined, these equations [29]
then yield $\delta T, \delta t_1$.

The iteration is continued until some convergence criterion
is satisfied. It must also be checked that w is a decreasing
function of time; actually all that is required is that
$w(t) > w(t_1)$ when $t < t_1$ and $w(t) < w(t_1)$ when $t > t_1$. It
should be pointed out that the relation involving the end con-
dition, equation [28], is only a necessary condition. There do
not seem to be any simple ways to ensure that a minimum, rather
than a maximum is obtained; in the case of constant gravity and
a ballistic target, the trajectory always furnishes a minimum.

6. Comments. The procedure seems to furnish a rather simple ex-
tension of the procedures which Bliss (3) introduced in Ballistics,
for calculating differentials. There seems to be only one

peculiarity of the trajectories, namely that the vector $\vec{w}$ is decreasing in magnitude.

If a minimum time of interception is desired, then set $t_1 = \text{lesser}(t_{1_{max}}, T)$. Equations [29] then furnish the changes $\delta T, \delta \Theta$. It is necessary to make the first estimate of Θ in a certain range, else the routine may not converge, or   may converge to the trajectory which takes maximum time (1,p 17).

The use of the acceleration rather than the mass as a variable simplifies the differential equations, since the mass enters non linearly in the differential equations. On the other hand, the bound on a is a function of time, not a constant, in the above example, and in more general cases it does not seem possible to express the bounds explicitly as functions of time.

There seem to be two general principles associated with optimum trajectories where there is a limited amount of energy available, so that throttling must occur in some form or another. The first of these is that the orientation of the energy-input vector, the acceleration in this case, is such as to maximize its projection onto the adjoint vector. The second is that the energy is to be expended at a time when the adjoint vector is large. The first condition above leads to the well-known steering criterion, apparently first published by Lawden [see (2) for references]. The second is apparently equivalent to the Weierstrass condition in some problems, though as expressed above, it has no analog in classic calculus of variations.

The method of solution, using procedures due to Bliss (3),(4), (5), has the nice feature that, should the actual trajectory deviate in some way from the planned trajectory, the same equations for the variations can be used to determine a corrective

thrust schedule. The vectors $\vec{w}_1, \vec{w}_2$ may be considered the

impulse response vectors for $x(T), y(T)$ respectively, since

the dot product $\vec{e} \cdot \vec{w}_i$ gives the change in the terminal value

of the corresponding variable due to a unit impulse in the

direction of $\vec{e}$.

In the computing which has been done, a modified Runge-

Kutta routine was used which has the desirable feature that it

is easy to change the time intervals, which must be done in the

neighborhood of t_1, T. No convergence problems have been en-

countered in the ballistic missile interception problem. On

the other hand it is apparently necessary to make a close initial

estimate in a corresponding orbital transfer problem. The values

obtained from the case where $\vec{g}$ varies at most linearly should

furnish reasonble starting values, should convergence be a problem.

Considerable attention has been attracted recently by the

maximum principle, due largely to papers by Pontriagin (6). To

the author this seems to be the fundamental way to approach prob-

lems in the calculus of variations;   the Euler equations

are derived   by effecting a stationary value for an integral

through the $\varepsilon\delta$ processes of calculus. In most problems there

seems to be no practical new information in the maximum prin-

ciple. Superficially the system of differential equations is

of lower order when the maximum principle is taken as the basic

concept. In the numerical solution, the last Euler equation [17]

may be solved by Newton's method, which depends on derivatives

so that it is equivalent to an additional differential equation.

Several authors have dealt with the mass and handled the

bounds on $\dot{m}$ by introducing a new variable, such as ϕ in the

equation

$$\dot{m} = \dot{m}_{max} \sin^2 \phi.$$

In simple problems, both the mathematical theory and the pro-
gramming of the numerical solution are simpler in terms of the
original variables; the logical decisions are of a type built
into all general-purpose digital computers.

It seems to the author that one of the most important
features in recent control theory rests on the use and inter-
pretation of the adjoint system of differential equations.
As remarked earlier, the foundations were laid by Bliss, with
more recent applications and contributions by Drenick (7),
Tsien (8), Breakwell (9), and Tyndall (10), among others. The
mathematical contribution to calculus of variations is in the
numerical procedures for solving two-(and more) point boundary-
value problems on high-speed computers. Another method of solu-
tion, the gradient method of Kelley (11) also utilizes the
adjoint system. It is sometimes stated that the adjoint equa-
tions [8] are necessary conditions for an extremal. This seems
a logical error: the variables u,v are introduced by the
mathematician, not implicit in the problem, and in turn are
chosen to satisfy [8], because that choice simplifies the inte-
gral in equation [7]. Dr. S. Ross usually brings out this fact.
In problems such as the above, the construction of the functional
of Pontriagin [(6), p 16] is equivalent to solving the three-
point problem involving the original system of equations and
the adjoint system.

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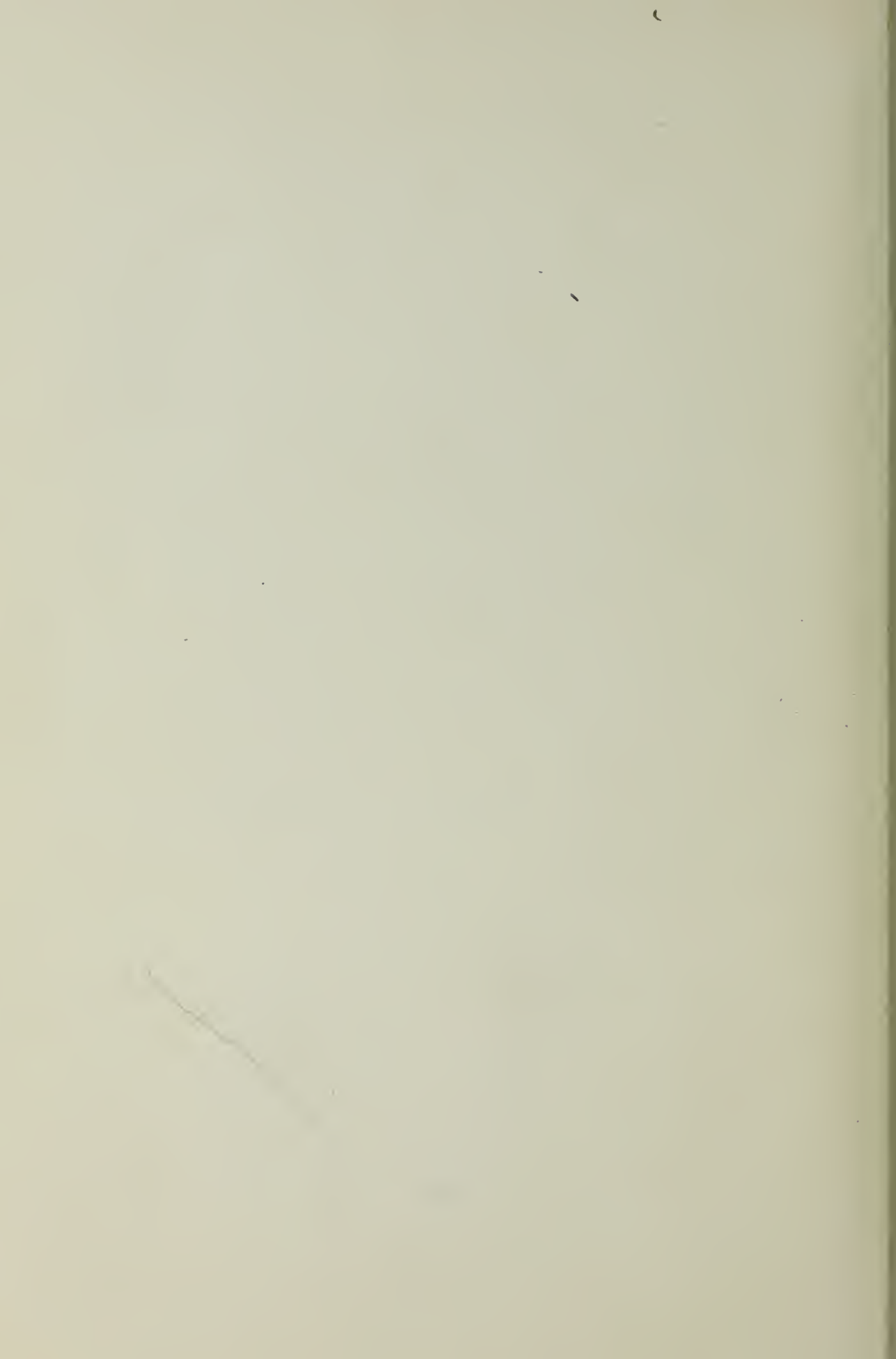Lecture Notes, The Martin Co., Denver, 1960.

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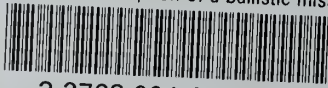Frank D. Faulkner
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